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A CHARACTERIZATION OF PLANAR GRAPHS BY TRÉMAUX ORDERS

H. DE FRAYSSEIX and P. ROSENSTIEHL

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A new characterization of planar graphs is stated in terms of an order relation on the vertices, called the Trémaux order, associated with any Trémaux spanning tree or Depth-First-Search Tree. The proof relies on the work of W. T. Tutte on the theory of crossings and the Trémaux algebraic theory of planarity developed by P. Rosenstiehl.

Introduction

Given a connected graph G, we denote by T a spanning tree of G, and by $T^{\perp}$ its complementary edge set, i.e. the cotree of G associated with T. We shall assume as obvious that there always exists an embedding of a graph in the plane, such that every pair of edges which cross is a pair of cotree edges having one crossing point exactly. Such an embedding is called below a *T-embedding* (see Fig. 1). The graph G is by definition a *planar graph* if and only if there exists a T-embedding of G in the plane with no crossing points.

For a given T-embedding $\tilde{G}$ we call *crossing pair,* any pair of edges which cross in $\tilde{G}$ and which are disjoint (i.e. with no common incident vertex). The latter restriction is justified by the following.

Tutte—Levow's crossing theorem [6] [3]: *Any T-embedding of a nonplanar graph has a crossing of two disjoint edges* (i.e. a crossing pair).

Given any vertex r of G, called the root, we shall consider a rooted tree (T, r) as a semi-lattice with lowest element r. It is convenient to say that every tree edge e is incident with a lower vertex $v^-(e)$ and with an upper vertex $v^+(e)$, which satisfy the relations:

$$r \leqq v^-(e) < v^+(e).$$

A *Trémaux tree* (or *Depth-First-Search tree*) of G is a rooted tree (T, r) such that every cotree edge $\alpha \in T^{\perp}$ is incident with two comparable vertices. It follows that every cotree edge α is incident with a lower vertex $v^-(e)$ and an upper

AMS subject classification (1980): 05 C 10, 57 M 15 vertex $v^+(e)$, which satisfy the relations:

$$r \leqq v^-(\alpha) \leqq v^+(\alpha)$$

(see Fig. 1).

The letter T below always refers to a Trémaux tree of a graph withouth loops.

For a cotree edge $\alpha \in T^\perp$, we denote by $T(\alpha)$ the unique chain of T having the same extremal incidences as the edge α (see Fig. 1). Y. Liu proved the following:

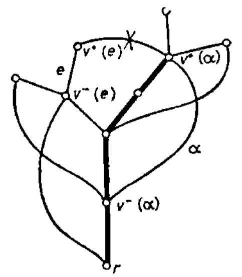

Fig. 1. Trémaux tree T and T-embedding, (tree edges e in straight lines, cotree edges α in curve lines, $T^+(\alpha)$ in heavy lines, the crossed edge is impossible)

Trémaux-crossing theorem ([4] [5]). *Given a Trémaux tree (T, r) of a non-planar graph G, any T-embedding of G has a crossing pair $[\alpha, \beta]$ such that $T(\alpha)$ and $T(\beta)$ have at least one edge in common.*

That theorem leads us to call *T-crossable pair* a pair of cotree edges $[\alpha, \beta]$ such that $T(\alpha)$ and $T(\beta)$ have at least one edge in common.

For a cotree edge $\alpha \in T^\perp$, we denote by $T^+(\alpha)$ the tree chain incident with r and $v^+(\alpha)$. For a given T-embedding $\tilde{G}$, every cotree edge α whose lower vertex is not the root, is said *embedded on the left side* (resp. *right side*), if the lower part of α is met on the left (resp. right) side, while the chain $T^+(\alpha)$ is followed from the root. Then two cotree edges of $\tilde{G}$ (whose lower vertex it not the root) are either *alike* or *opposite,* relatively to their side of embedding.

In § 1 we show that the semi-lattice order of a Trémaux tree T allows us to define, in an abstract way, T-alike or T-opposite pairs of cotree edges (whose lower vertex is not the root). Geometrically that means that, in any embedding of G without crossings, T-alike edges are necessarily embedded on the same side, and T-opposite edges are necessarily embedded on opposite sides. A characterization of planar graphs follows in § 2, with proof in § 3.

1. Abstract T-alike and T-opposite relations

Let us define the T-alike relation and the T-opposite relation on the set of cotree edges of a graph G with a Trémaux tree (T, r). For that, we display edge patterns where the status of two cotree edges is determined from the order relation within the incident vertices of three or four edges. There are three types of patterns to be considered.

Type (i): Whenever three cotree edges $\alpha, \beta, \gamma \in T^{\perp}$ are such that

$$v^{-}(\gamma) < v^{-}(\alpha) \leqq v^{-}(\beta) < v^{+}(\alpha) \wedge v^{+}(\beta) \wedge v^{+}(\gamma) < v^{+}(\alpha) \wedge v^{+}(\beta),$$

we say that α and β are *T-alike.*

On Fig. 2-(i) all possible patterns belonging to type (i) are symbolically represented. It has to be understood that to any strict inequality corresponds a tree edge of the figure. The large circles represent a subtree of (T, r) which may eventually be reduced to a vertex.

It is easy to check that in any T-embedding without crossings of a graph, the edges α and β of a pattern of type (i) are necessarily embedded on the same side.

Type (ii): Whenever three cotree edges $\alpha, \beta, \gamma \in T^{\perp}$ are such that

$$v^{-}(\gamma) < v^{-}(\alpha) < v^{-}(\beta) < v^{+}(\alpha) \wedge v^{+}(\beta) \wedge v^{+}(\gamma) < v^{+}(\beta) \wedge v^{+}(\gamma),$$

we say that α and β are *T-opposite.*

On Fig. 2-(ii) all patterns belonging to type (ii) are symbolically represented. It is easy to check that in any T-embedding without crossings of a graph, the edges α and β of a pattern of type (ii) are necessarily embedded on opposite sides.

Type (iii): Whenever four cotree edges $\alpha, \beta, \gamma, \delta \in T^{\perp}$ are such that

$$v^{-}(\gamma) = v^{-}(\delta) < v^{-}(\alpha) = v^{-}(\beta) < v^{+}(\alpha) \wedge v^{+}(\beta),$$

$$v^{+}(\alpha) \wedge v^{+}(\beta) < v^{+}(\alpha) \wedge v^{+}(\gamma),$$

$$v^{+}(\alpha) \wedge v^{+}(\beta) < v^{+}(\beta) \wedge v^{+}(\delta),$$

we say that α and β are *T-opposite.*

On Fig. 2-(iii) all patterns belonging to type (iii) are symbolically represented.

It is easy to check that in any T-embedding without crossings of a graph, the edges α and β of a pattern of type (iii) are necessarily embedded on opposite sides.

Let us notice that cubic graphs have no patterns of type (iii).

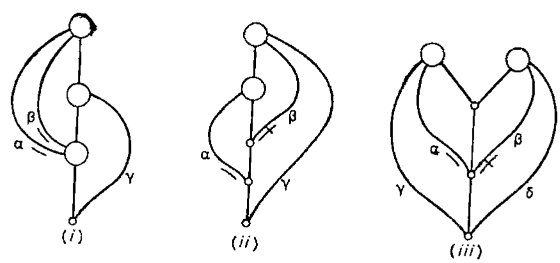

Fig. 2. The tree types of patterns which define T-alike and T-opposite relations, (α and β are T-alike in (i), T-opposite in (ii) and (iii))

2. The characterization theorem

Theorem. *A connected graph G, with a Trémaux tree (T, r) is planar if there exists a partition of its cotree edges into two classes, such that two edges which are T-alike belong to the same class, and two edges which are T-opposite belong to different classes.*

Fig. 3.1 displays for $K_{3,3}$ two edges β and γ which are alike and opposite at the same time.

Fig. 3.2 displays for K_5 three edges β, γ, δ with γ alike β, β alike δ and also γ opposite to δ.

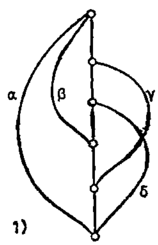

Fig. 3.1: $K_{3,3}$
$\beta, \gamma, \delta \xrightarrow{(i)} \beta, \gamma$ T-alike
$\alpha, \beta, \gamma \xrightarrow{(ii)} \beta, \gamma$ T-opposite

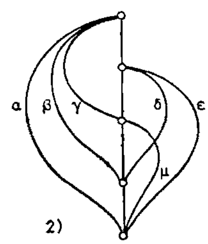

Fig. 3.2: K_5
$\beta, \gamma, \varepsilon \xrightarrow{(i)} \beta, \gamma$ T-alike
$\alpha, \gamma, \delta \xrightarrow{(ii)} \gamma, \delta$ T-opposite
$\beta, \delta, \mu, \xrightarrow{(i)} \beta, \delta$ T-alike

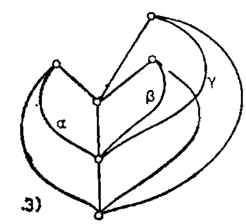

Fig. 3.3: A non-planar graph α, β, γ should be pairwise T-opposite according to patterns (iii)

We notice that patterns of types (i) and (ii) appear in both graph $K_{3,3}$ and K_5.

In order to illustrate the necessity of case (iii) for a characterization of planarity, it can be observed that in the graph G of Fig. 3.3, case (i) and (ii) do not appear, although G is a non-planar graph.

Last of all, Fig. 4 displays a T-embedding $\tilde{G}$ of a planar graph where every pair of T-alike edges are embedded on the same side, and every pair of T-opposite edges are embedded on opposite sides, and nevertheless $\tilde{G}$ has a crossing T-crossable pair $[\beta, \gamma]$.

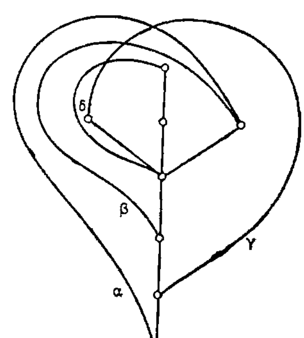

Fig. 4. A good T-embedding $\tilde{G}$ $[\gamma, \delta]$ is a crossing non T-crossable pair. β and γ are T-opposite, and although β and γ have their lower ends embedded on opposite sides, $[\beta, \gamma]$ is a crossing T-crossable pair

3. Proof of the theorem

Let us consider some additional definitions related to a Trémaux tree (T, r).

For a tree edge $e \in T$, let us denote by $T^{\perp}(e)$ the set of cotree edges α such that

$$v^-(\alpha) \leqq v^-(e) < v^+(e) \leqq v^+(\alpha).$$

Given two tree edges $e, f \in T$ with $v^-(e) = v^-(f)$, we call *switching set* $\mu(e, f)$ the set of T-crossable pairs occurring in the cartesian product of $T^{\perp}(e)$ and $T^{\perp}(f)$.

It is obvious that whenever a T-crossable pair $[\alpha, \beta]$ is such that $v^+(\alpha)$ and $v^+(\beta)$ are not comparable, $[\alpha, \beta]$ belongs to exactly one switching set $\mu(e, f)$ (see Fig. 5).

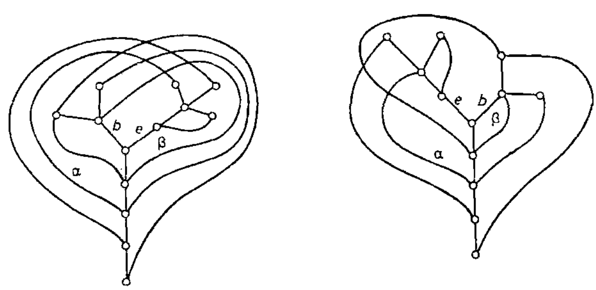

Fig. 5. Switching pair (e, f) involves the switching set $\mu(e, f)$. (By the switching, crossing pairs $\mu(e, f)$ become non-crossing pairs and conversely)

The consideration for the proof only of pairs which are T-crossable is justified by the Trémaux-crossing theorem.

The formalization of the proof of the theorem of the paper relies on the following theorem which is a weak form of the Trémaux algebraic planarity characterization by P. Rosenstiehl [5].

Switching theorem. *Given a connected graph G, a Trémaux tree (T, r) of G, and a T-embedding $\tilde{G}$, G is planar if the set of crossing T-crossable pairs of $\tilde{G}$ may be partitioned into switching sets of G.*

Let G be a graph which satisfies the condition of our theorem: the cotree edges are then partitionned into two classes compatible with the T-alike and T-opposite relations. We call a *good T-embedding* of G, a T-embedding of G such that T-alike edges are embedded on the same side, and T-opposite edges are embedded on opposite sides, and moreover, such that the two following conditions EMBED 1 and EMBED 2 are satisfied.

EMBED 1: Any T-crossable pair $[\alpha, \beta]$ such that

$$v^-(\alpha) < v^-(\beta) < v^+(\alpha) < v^+(\beta)$$

is not a crossing pair.

Geometrically EMBED 1 condition means that in a good T-embedding the upper part of α and the lower part of β are embedded on opposite sides, relatively to the chain $T^+(\beta)$.

EMBED 2: Any T-crossable pair $[\alpha, \beta]$ such that

$$v^-(\beta) < v^-(\alpha) < v^+(\alpha) < v^+(\beta)$$

is not a crossing pair.

Geometrically EMBED 2 condition means that in a good T-embedding the upper part and lower part of the edge α are embedded on the same side, relatively to the chain $T^+(\beta)$.

Starting with an arbitrary embedding of the Trémaux tree T one can construct a good T-embedding $\tilde{G}_0$, by adding the cotree edges one by one, choosing the side of embedding of the lower part of each edge according to the class to which it belongs. Furthermore, for any crossable pair $[\alpha, \beta]$ involved in conditions EMBED 1 or EMBED 2, by choosing the side of embedding of the upper part of α such that $[\alpha, \beta]$ is not a crossing pair. No contradiction arises during the construction as we check it by inspecting below the three cases for two T-crossable pairs $[\alpha, \beta]$ and $[\alpha, \beta']$ being constrained by EMBED 1 and/or EMBED 2.

(a) If $[\alpha, \beta]$ and $[\alpha, \beta']$ are both constrained by EMBED 1 condition, the edges α, β, β' form a pattern of type (i) where β and β' are T-alike, then β and β' are embedded on the same side.
(b) If $[\alpha, \beta]$ and $[\alpha, \beta']$ are both constrained by EMBED 2 condition, it follows that the upper part and lower part of α are embedded on the same side.
(c) If $[\alpha, \beta]$ is constrained by EMBED 1 condition, and $[\alpha, \beta']$ is constrained by EMBED 2 condition, the edges α, β, β' form a pattern of type (ii) where α and β are T-opposite, and then the upper part and lower part of α are on the same side.

As we have assumed that $\tilde{G}_0$ is a good T-embedding, each crossing pair of $\tilde{G}_0$ is a T-crossable pair $[\alpha, \beta]$ for which $v^+(\alpha)$ and $v^+(\beta)$ are not comparable, and therefore belongs to a unique switching set $\mu(e, f)$. In order to apply the Switching Theorem we prove now that whenever a pair of a switching set $\mu(e, f)$ is a crossing pair, all the pairs of $\mu(e, f)$ are crossing pairs. What means geometrically the Switching theorem, is that we can eliminate the crossing set $\mu(e, f)$ of $\tilde{G}_0$ by *switching* the pair of tree edges (e, f).

Let us consider a crossing pair $[\alpha, \beta]$ of $\tilde{G}_0$ belonging to the switching set $\mu(e, f)$ and another pair $[\alpha', \beta']$ also belonging to $\mu(e, f)$. We shall prove that α' and β' do cross.

We assume first the hypothesis of α and β being T-opposite. Without loss of generality we suppose that $v^-(\alpha) < v^-(\beta)$. Three types of cases appear for the relative positions of $[\alpha, \beta]$ and $[\alpha', \beta']$.

(A) $v^-(\beta') > v^-(\alpha)$ (see Fig. 6.1 and 6.2)

(1) if $v^-(\alpha') > v^-(\beta')$, then α, β, β' is a pattern of type (i) and α, α', β' is a pattern of type (ii.) Hence α' and β are T-opposite.

(2) if $v^-(\alpha') < v^-(\beta')$, then α, β, β' and α', β, β' form two patterns of type (i). Hence α and β' are T-opposite.

(B) $v^-(\beta')=v^-(\alpha)$ (see Fig. 7.1, 7.2, 7.3 and 7.4)

(1) if $v^-(\alpha')=v^-(\beta)$, then $\alpha, \beta, \alpha', \beta'$ form a symmetrical pattern of type (iii). Hence α' and β are T-opposite.

(2) if $v^-(\alpha')>v^-(\beta)$, then α, α', β form a pattern of type (ii). Hence α' and β are T-opposite.

(3) if $v^-(\beta')<v^-(\alpha')<v^-(\beta)$, then α', β, β' form a pattern of type (ii). Hence α' and β are T-opposite.

(4) if $v^-(\alpha')<v^-(\beta')$, then α', β, β' form a pattern of type (ii). Hence α and β' are T-opposite.

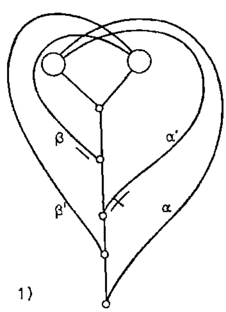

Fig. 6.1: $v^-(\alpha')>v^-(\beta')$
(α' and β are T-opposite)

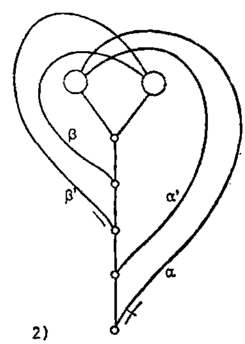

Fig. 6.2: $v^-(\alpha')<v^-(\beta')$
(α and β' are T-opposite)

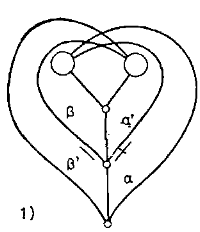

Fig. 7.1: $v^-(\alpha')=v^-(\beta)$
(α' and β are T-opposite)

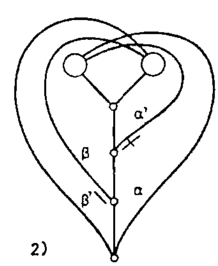

Fig. 7.2: $v^-(\alpha')>v^-(\beta)$
(α' and β are T-opposite)

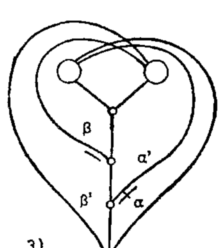

Fig. 7.3: $v^-(\beta')<v^-(\alpha')<v^-(\beta)$
(α' and β are T-opposite)

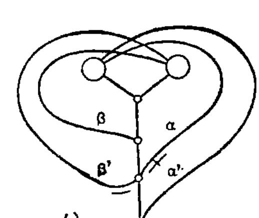

Fig. 7.4: $v^-(\alpha')<v^-(\beta')$
(α and β' are T-opposite)

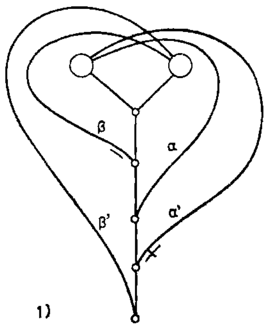

Fig. 8.1: $v^-(\alpha') > v^-(\beta')$
(α' and β are T-opposite)

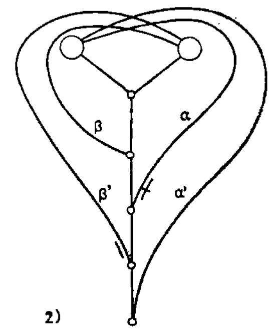

Fig. 8.2: $v^-(\alpha') < v^-(\beta')$
(α and β' are T-opposite)

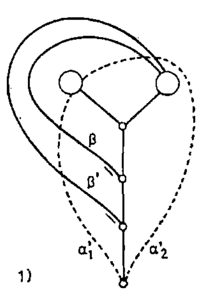

Fig. 9.1. α, β' T-opposite.
(whatever is the side of embedding of α'— α_1' or α_2'—α' crosses β')

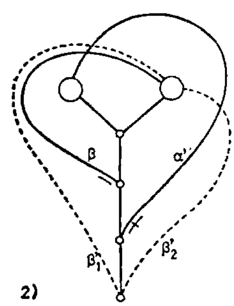

Fig. 9.2. α', β T-opposite.
(whatever is the side of embedding of β'— β_1' or β_2'—α' crosses β')

(C) $v^-(\beta') < v^-(\alpha)$ (see Fig. 8.1 and 8.2)

(1) if $v^-(\alpha') > v^-(\beta')$, then α, β, β' is a pattern of type (ii) and α, α', β' is a pattern of type (ii). Hence α' and β are T-opposite.

(2) if $v^-(\alpha') < v^-(\beta')$, then α, α', β' and α, β, β' form two patterns of type (ii). Hence α and β' are T-opposite.

Actually in all cases above we have either $v^-(\alpha') < v^-(\beta')$ and α, β' T-opposite, or $v^-(\beta') < v^-(\alpha')$ and α', β T-opposite. Then α' crosses β' as α crosses β, (see Fig. 9.1 and 9.2).

The hypothesis of α and β being T-alike should be considered. But all the possible cases generate similar conclusions through similar arguments.

At last, the crossing pairs of $\tilde{G}_0$ being partitioned into switching sets, the Switching theorem is applied to $\tilde{G}_0$; and G is planar.

Remark. The above characterization theorem can be used as a topological tool to justify planarity testing algorithms involving Depth-First-Search, such as the famous and first one in the literature due to J. Hopcroft and R. E. Tarjan [2], (for more on the subject see S. G. Williamson [7]), or our Left-Right algorithm dealing straighforwardly with the T-alike and T-opposite relations [1]. The main difficulty in that matter is to find a global argument when the algorithm fails. Although the T-alike and T-opposite relations are quite different from the relations which operate in the Hopcroft—Tarjan algorithm, their transitive closures are both the same; and hence the same theorem could be used as a topological argument in the proof.

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H. de Fraysseix, P. Rosenstiehl

EHESS, 54, Bd Raspail
75270 Paris Cédex 06
France